A Primal-Dual Algorithm for Convex Nonsmooth Optimization on Riemannian Manifolds

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joint work with
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The Model

We consider a minimization problem

$$\operatorname*{arg\,min}_{p\in\mathcal{C}} F(p) + G(\Lambda(p))$$

- $\mathcal{M},\mathcal{N}$ are (high-dimensional) Riemannian Manifolds
- $F\colon \mathcal{M} \to \overline{\mathbb{R}}$ nonsmooth, (locally, geodesically) convex
- $G\colon \mathcal{N} \to \overline{\mathbb{R}}$ nonsmooth, (locally) convex
- $\Lambda\colon \mathcal{M} \to \mathcal{N}$ nonlinear
- $\mathcal{C} \subset \mathcal{M}$ strongly geodesically convex.

➔ In image processing:
choose a model, such that finding a minimizer yields the reconstruction

Splitting Methods & Algorithms

On a Riemannian manifold $\mathcal{M}$ we have

- Cyclic Proximal Point Algorithm (CPPA) [Bačák 2014]
- (parallel) Douglas–Rachford Algorithm (PDRA) [RB, Persch, and Steidl 2016]

On $\mathbb{R}^n$ PDRA is known to be equivalent to [O'Connor and Vandenberghe 2018; Setzer 2011]

- Primal-Dual Hybrid Gradient Algorithm (PDHGA) [Esser, Zhang, and Chan 2010]
- Chambolle-Pock Algorithm (CPA) [Chambolle and Pock 2011; Pock, Cremers, Bischof, and Chambolle 2009]

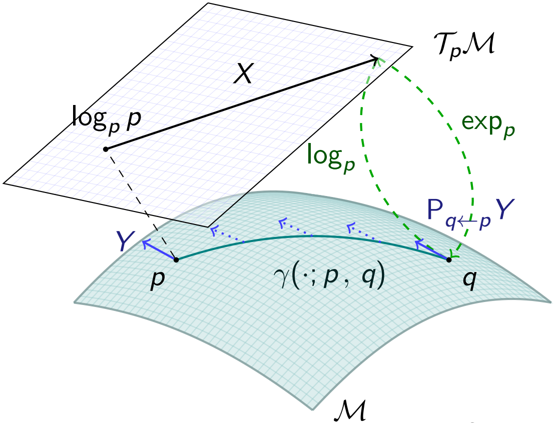
But on a Riemannian manifold $\mathcal{M}$: ⚠ no duality theory!

Goals of this talk.

Formulate Duality on a Manifold

Derive a Riemannian Chambolle–Pock Algorithm (RCPA)

A d-dimensional Riemannian manifold $\mathcal{M}$



A d-dimensional Riemannian manifold can be informally defined as a set $\mathcal{M}$ covered with a 'suitable' collection of charts, that identify subsets of $\mathcal{M}$ with open subsets of $\mathbb{R}^d$ and a continuously varying inner product on the tangent spaces.

[Absil, Mahony, and Sepulchre 2008]

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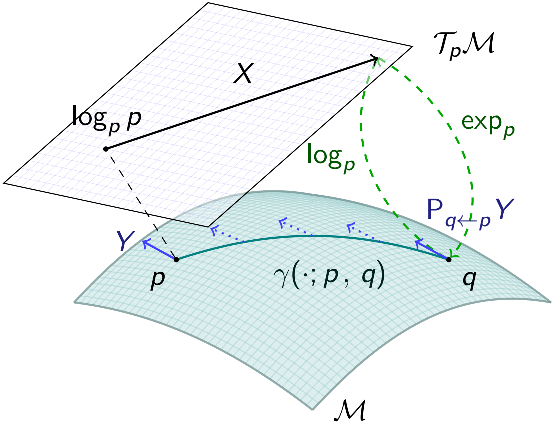
A d-dimensional Riemannian manifold $\mathcal{M}$



Geodesic $\gamma(\cdot; p, q)$
a shortest path between $p, q \in \mathcal{M}$

Tangent space $\mathcal{T}_p\mathcal{M}$ at p
with inner product $(\cdot, \cdot)_p$

Logarithmic map $\log_p q = \dot\gamma(0; p, q)$
"speed towards q"

Exponential map $\exp_p X = \gamma_{p,X}(1)$,
where $\gamma_{p,X}(0) = p$ and $\dot\gamma_{p,X}(0) = X$

Parallel transport $\mathrm{P}_{q\leftarrow p}Y$
from $\mathcal{T}_p\mathcal{M}$ along $\gamma(\cdot; p, q)$ to $\mathcal{T}_q\mathcal{M}$

Musical Isomorphisms

[Lee 2003]

The dual space $\mathcal{T}_p^*\mathcal{M}$ of a tangent space $\mathcal{T}_p\mathcal{M}$ is called cotangent space.
We denote by $\langle\cdot\,,\cdot\rangle$ the duality pairing.

We define the musical isomorphisms

- $\flat\colon \mathcal{T}_p\mathcal{M} \ni X \mapsto X^\flat \in \mathcal{T}_p^*\mathcal{M}$ via $\langle X^\flat\,, Y\rangle = (X, Y)_p$ for all $Y \in \mathcal{T}_p\mathcal{M}$
- $\sharp\colon \mathcal{T}_p^*\mathcal{M} \ni \xi \mapsto \xi^\sharp \in \mathcal{T}_p\mathcal{M}$ via $(\xi^\sharp\,, Y)_p = \langle\xi\,, Y\rangle$ for all $Y \in \mathcal{T}_p\mathcal{M}$.

$\Rightarrow$ inner product and parallel transport on/between $\mathcal{T}_p^*\mathcal{M}$

Convexity

[Sakai 1996; Udriște 1994]

A set $\mathcal{C} \subset \mathcal{M}$ is called (strongly geodesically) convex
if for all $p, q \in \mathcal{C}$ the geodesic $\gamma(\cdot; p, q)$ is unique and lies in $\mathcal{C}$.

A function $F\colon \mathcal{C} \to \overline{\mathbb{R}}$ is called (geodesically) convex
if for all $p, q \in \mathcal{C}$ the composition $F(\gamma(t; p, q)), t \in [0, 1]$, is convex.

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The Euclidean Fenchel Conjugate

Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper and convex.
We define the Fenchel conjugate $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ of f by

$$f^*(\xi) := \sup_{x\in\mathbb{R}^n} \langle \xi, x\rangle - f(x) = \sup_{x\in\mathbb{R}^n} \begin{pmatrix} \xi \\ -1 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} x \\ f(x) \end{pmatrix}$$

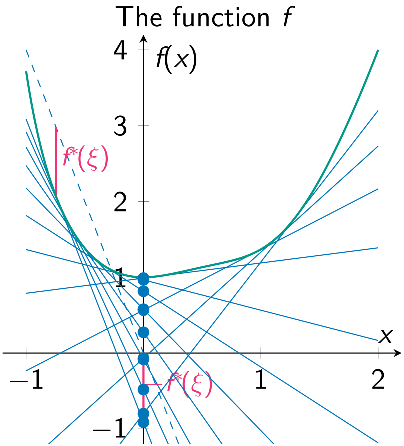
- interpretation: maximize the distance of $\xi^{\mathsf{T}}x$ to f

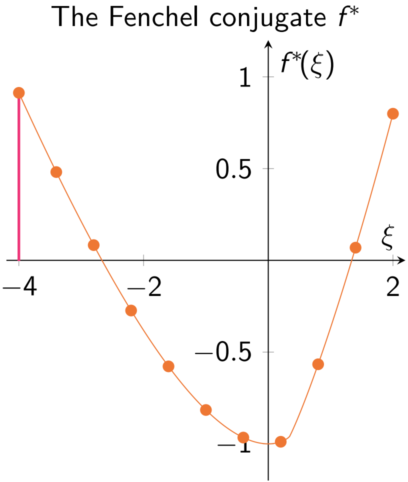
$\Rightarrow$ extremum seeking problem on the epigraph

The Fenchel biconjugate reads

$$f^{**}(x) = (f^*)^*(x) = \sup_{\xi\in\mathbb{R}^n} \{\langle \xi , x\rangle - f^*(\xi)\}.$$

Illustration of the Fenchel Conjugate





The Riemannian $m-$Fenchel Conjugate

[RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]
alternative approach: [Ahmadi Kakavandi and Amini 2010]

Idea: Introduce a point on $\mathcal{M}$ to "act as" 0.
Let $m \in \mathcal{C} \subset \mathcal{M}$ be given and $F\colon \mathcal{C} \to \overline{\mathbb{R}}$.
The m-Fenchel conjugate $F_m^*\colon \mathcal{T}_m^*\mathcal{M} \to \overline{\mathbb{R}}$ is defined by

$$F_m^*(\xi_m) \coloneqq \sup_{X \in \mathcal{L}_{\mathcal{C},m}} \bigl\{\langle \xi_m\,, X\rangle - F(\exp_m X)\bigr\},$$

where $\mathcal{L}_{\mathcal{C},m} \coloneqq \{X \in \mathcal{T}_m\mathcal{M} \mid q = \exp_m X \in \mathcal{C} \text{ and } \|X\|_p = d(q,p)\}$.

Let $m' \in \mathcal{C}$.
The mm'-Fenchel-biconjugate $F_{mm'}^{**}\colon \mathcal{C} \to \overline{\mathbb{R}}$ is given by

$$F_{mm'}^{**}(p) = \sup_{\xi_{m'} \in \mathcal{T}_{m'}^*\mathcal{M}} \bigl\{\langle \xi_{m'}\,, \log_{m'} p\rangle - F_m^*(\mathrm{P}_{m \leftarrow m'}\xi_{m'})\bigr\}.$$

usually we only use the case $m = m'$.

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Saddle Point Formulation

Let F be geodesically convex, $G \circ \exp_n$ be convex (on $\mathcal{T}_n\mathcal{N}$).

From

$$\min_{p \in \mathcal{C}} F(p) + G(\Lambda(p))$$

we derive the saddle point formulation for the n-Fenchel conjugate of G as

$$\min_{p \in \mathcal{C}} \max_{\xi_n \in \mathcal{T}_n^*\mathcal{N}} \langle \xi_n , \log_n \Lambda(p) \rangle + F(p) - G_n^*(\xi_n).$$

But $\Lambda \colon \mathcal{M} \to \mathcal{N}$ is a non-linear operator!

For Optimality Conditions and the Dual Prolem: What's Λ^*?

[Valkonen 2014]

Approach. Linearization:

$$\Lambda(p) \approx \exp_{\Lambda(m)} D\Lambda(m)[\log_m p]$$

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The exact Riemannian Chambolle–Pock Algorithm (eRCPA)

Input: $m, p^{(0)} \in \mathcal{C} \subset \mathcal{M},\ n = \Lambda(m),\ \xi_n^{(0)} \in \mathcal{T}_n^*\mathcal{N},$
and parameters $\sigma,\ \tau,\ \theta > 0$

$k \leftarrow 0$

$\bar{p}^{(0)} \leftarrow p^{(0)}$

while not converged **do**

$\quad \xi_n^{(k+1)} \leftarrow \operatorname{prox}_{\tau G_n^*} \xi_n^{(k)} + \tau\bigl(\log_n \Lambda(\bar{p}^{(k)})\bigr)^\flat$

$\quad p^{(k+1)} \leftarrow \operatorname{prox}_{\sigma F} \exp_{p^{(k)}}\Bigl(\mathrm{P}_{m\leftarrow p^{(k)}}\bigl(-\sigma D\Lambda(m)^*[\xi_n^{(k+1)}]\bigr)^\sharp\Bigr)$

$\quad \bar{p}^{(k+1)} \leftarrow \exp_{p^{(k+1)}}\bigl(-\theta \log_{p^{(k+1)}} p^{(k)}\bigr)$

$\quad k \leftarrow k+1$

end while

Output: $p^{(k)}$

Generalizations & Variants of the RCPA

Classically

[Chambolle and Pock 2011]

- change $\sigma = \sigma_k$, $\tau = \tau_k$, $\theta = \theta_k$ during the iterations
- introduce an acceleration γ
- relax dual $\bar{\xi}$ instead of primal $\bar{p}$ (switches lines 4 and 5)

Furthermore we

[RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

- introduce the IRCPA: linearize Λ, i. e., adopt the Euclidean case from [Valkonen 2014]

$$\log_n \Lambda(\bar{p}^{(k)}) \quad \to \quad \mathrm{P}_{n \leftarrow \Lambda(m)} D\Lambda(m)[\log_m \bar{p}^{(k)}]$$

- choose $n \neq \Lambda(m)$ introduces a parallel transport

$$D\Lambda(m)^*[\xi_n^{(k+1)}] \quad \to \quad D\Lambda(m)^*[\mathrm{P}_{\Lambda(m) \leftarrow n} \xi_n^{(k+1)}]$$

- change $m = m^{(k)}$, $n = n^{(k)}$ during the iterations

The Linearized RCPA with Dual Relaxation

We introduce for ease of notation

$$\widetilde{p}^{(k)} = \exp_{p^{(k)}}\Bigl(\mathrm{P}_{p^{(k)}\leftarrow m} - \bigl(\sigma\bigl(D\Lambda(m)\bigr)^{*}[\bar{\xi}_n^{(k)}]\bigr)^{\sharp}\Bigr)$$

for the linearized Riemannian Chambolle Pock
with dual relaxed

$$\bar{\xi}_n^{(k)} \leftarrow \xi_n^{(k)} + \theta(\xi_n^{(k)} - \xi_n^{(k-1)}).$$

Especially for $\theta = 1$ we obtain

$$\bar{\xi}_n^{(k)} = 2\xi_n^{(k)} - \xi_n^{(k-1)}.$$

A Conjecture

We define

$$C(k) := \frac{1}{\sigma} d^2(p^{(k)}, \tilde{p}^{(k)}) + \langle \bar{\xi}_n^{(k)}, D\Lambda(m)[\zeta_k] \rangle,$$

where

$$\zeta_k = \mathrm{P}_{m \leftarrow p^{(k)}} \big(\log_{p^{(k)}} p^{(k+1)} - \mathrm{P}_{p^{(k)} \leftarrow \tilde{p}^{(k)}} \log_{\tilde{p}^{(k)}} \widehat{p} \big) - \log_m p^{(k+1)} + \log_m \widehat{p},$$

and $\hat{p}$ is a minimizer of the primal problem.

Remark.
For $\mathcal{M} = \mathbb{R}^d$: $\zeta_k = \tilde{p}^{(k)} - p^{(k)} = -\sigma(D\Lambda(m))^*[\bar{\xi}_n^{(k)}] \Rightarrow C(k) = 0$.

Conjecture.
Assume $\sigma\tau < \|D\Lambda(m)\|^2$. Then $C(k) \geq 0$ for all $k > K$, $K \in \mathbb{N}$.

Convergence of the IRCPA

Theorem. [RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

Let $\mathcal{M}$, $\mathcal{N}$ be Hadamard. Assume that the linearized problem

$$\min_{p\in\mathcal{M}} \max_{\xi_n\in\mathcal{T}_n^*\mathcal{N}} \langle (D\Lambda(m))^*[\xi_n]\,, \log_m p\rangle + F(p) - G_n^*(\xi_n).$$

has a saddle point $(\widehat{p}, \widehat{\xi}_n)$.
Choose σ, τ such that

$$\sigma\tau < \lVert D\Lambda(m)\rVert^2$$

and assume that $C(k) \geq 0$ for all $k > K$. Then it holds

1. the sequence $(p^{(k)}, \xi_n^{(k)})$ remains bounded,
2. there exists a saddle-point (p', ξ_n') such that $p^{(k)} \to p'$ and $\xi_n^{(k)} \to \xi_n'$.

The ℓ^2-TV Model

[Rudin, Osher, and Fatemi 1992; Lellmann, Strekalovskiy, Koetter, and Cremers 2013; Weinmann, Demaret, and Storath 2014]

For a manifold-valued image $f \in \mathcal{M}$, $\mathcal{M} = \mathcal{N}^{d_1,\ d_2}$, we compute

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} \frac{1}{\alpha} F(p) + G(\Lambda(p)), \qquad \alpha > 0,$$

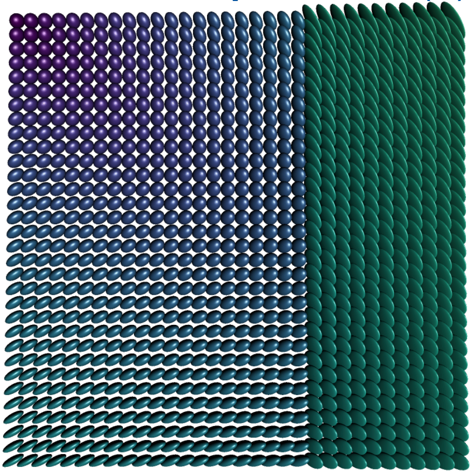
with

- data term $F(p) = \frac{1}{2} d^2_{\mathcal{M}}(p, f)$
- “forward differences” $\Lambda\colon \mathcal{M} \to (T\mathcal{M})^{d_1-1,\ d_2-1,\ 2}$,

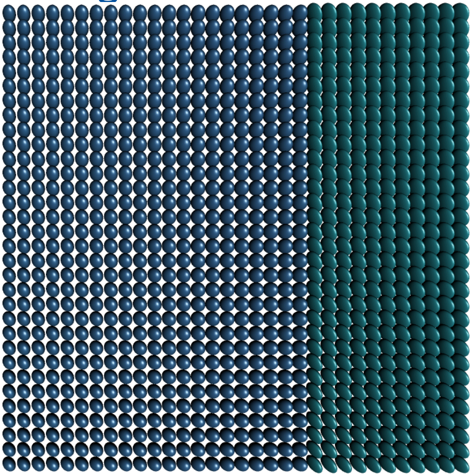
$$p \mapsto \Lambda(p) = \Bigl(\bigl(\log_{p_i} p_{i+e_1},\ \log_{p_i} p_{i+e_2}\bigr) \Bigr)_{i\in\{1,\ldots,d_1-1\}\times\{1,\ldots,d_2-1\}}$$

- prior $G(X) = \lVert X \rVert_{g,q,1}$ similar to a collaborative TV [Duran, Moeller, Sbert, and Cremers 2016]

Numerical Example for a $\mathcal{P}(3)$-valued Image



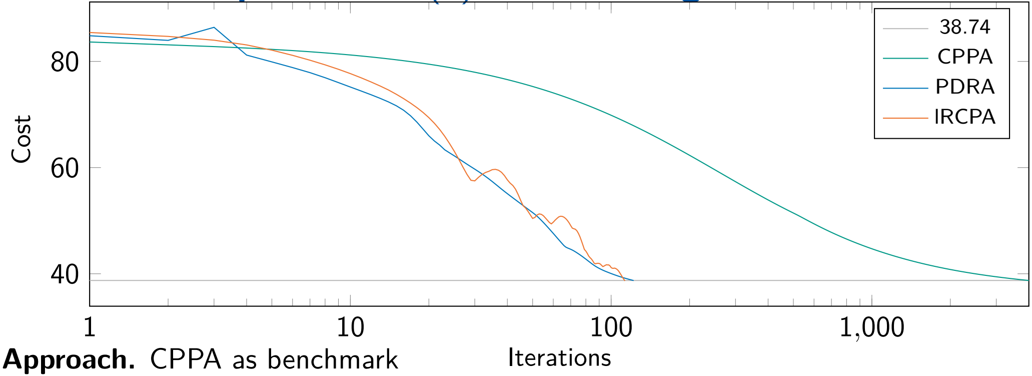
$\mathcal{P}(3)$-valued data.



anisotropic TV, $\alpha = 6$.

- in each pixel we have a symmetric positive definite matrix
- Applications: denoising/inpainting e.g. of DT-MRI data

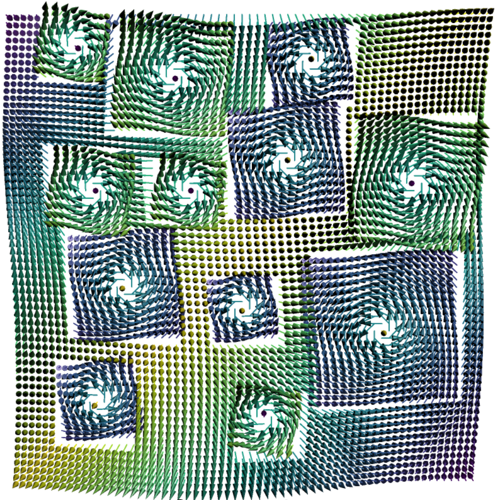
Numerical Example for a $\mathcal{P}(3)$-valued Image



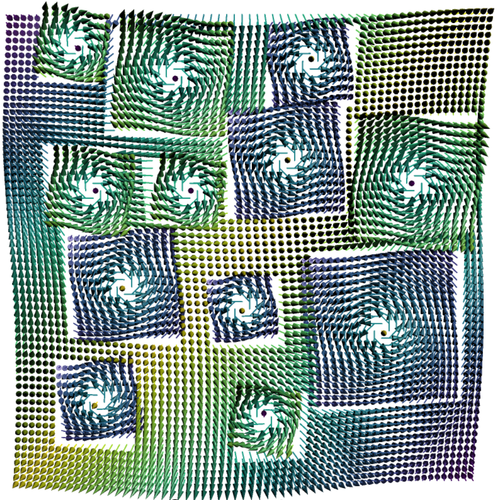
Approach. CPPA as benchmark

	CPPA	PDRA	IRCPA
parameters	$\lambda_k = \frac{4}{k}$	$\eta = 0.58$ $\lambda = 0.93$	$\sigma = \tau = 0.4$ $\gamma = 0.2,\ m = I$
iterations	4000	122	**113**
runtime	1235 s.	380 s.	**96.1 s.**

Base point Effect on $\mathbb{S}^2$-valued data

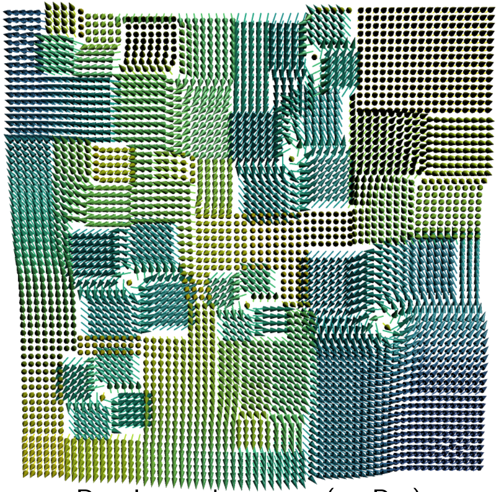


Original data

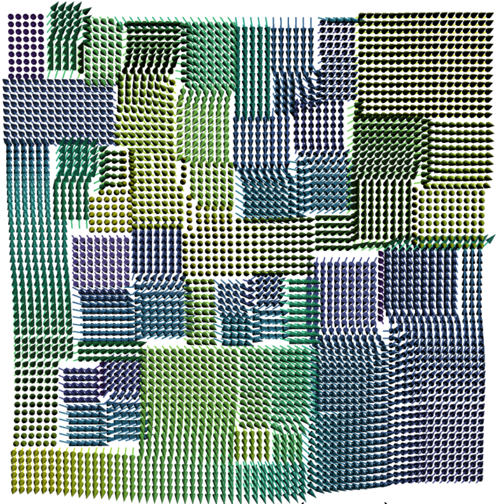


Original data

Base point Effect on $\mathbb{S}^2$-valued data

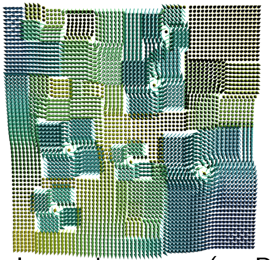


Result, m the mean (p. Px.)



Result, m west (p. Px.)

Base point Effect on $\mathbb{S}^2$-valued data



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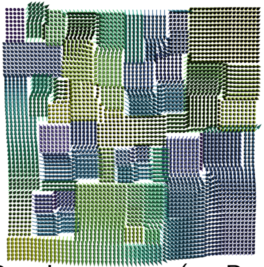

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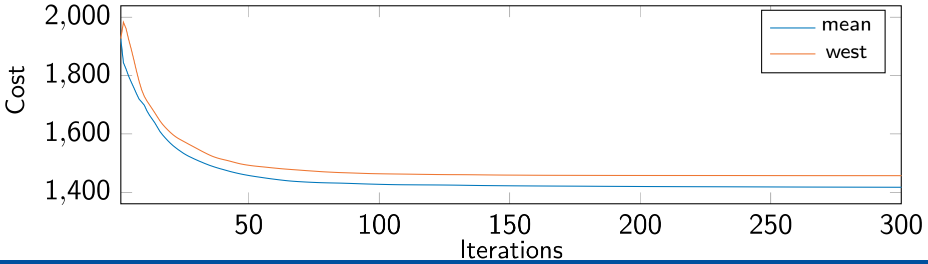


Summary & Outlook

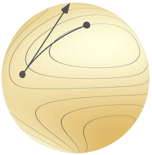
Summary.

- We introduced a duality framework on Riemannian manifolds
- We derived a Riemannian Chambolle–Pock Algorithm
- Numerical examples illustrate performance

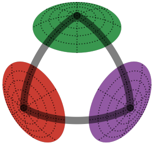
Outlook.

- investigate $C(k)$ and the error of linearization
- strategies for choosing m, n (adaptively)
- alternative models of Fenchel duality (e. g. without m) [RB, Herzog, and Silva Louzeiro 2021]
  Thu @ 11:00 BST (18:00 CEST) in MS Optimization and Manifolds
- higher order methods non-smooth methods [Diepeveen and Lellmann 2021]
  W. Diepeveen, Thu @ 11:30 BST (18:30 CEST) in MS Optimization and Manifolds

Reproducible Research



The algorithm is published in `Manopt.jl`, a Julia Package available at `http://manoptjl.org`.



It uses the interface from `ManifoldsBase.jl` and any manifold from `Manifolds.jl` can be used in the algorithms.
`https://juliamanifolds.github.io/Manifolds.jl/`
[Axen, Baran, RB, and Rzecki 2021]

Goal.

Being able to use an(y) algorithm for a(ny) model directly on a(ny) manifold easily and efficiently.

Alternatives.

- Manopt, manopt.org (Matlab, by N. Boumal)
- pymanopt, pymanopt.github.io (Python, by S. Weichwald, J. Townsend, N. Koep)

Selected References

Ahmadi Kakavandi, B. and M. Amini (Nov. 2010). "Duality and subdifferential for convex functions on complete metric spaces". In: *Nonlinear Analysis: Theory, Methods & Applications* 73.10, pp. 3450–3455. DOI: 10.1016/j.na.2010.07.033.

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