Second Order Differences of Cyclic Data and Applications in Variational Denoising

Ronny Bergmann*

Image Processing Group
Departement of Mathematics
University of Technology Kaiserslautern

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FELIX KLEIN
ZENTRUM FÜR
MATHEMATIK

*joint work with A. Weinmann, G. Steidl, F. Laus

Outline

Introduction

- Employing the Rudin-Osher-Fatemi (ROF) functional [Osher, Rudin, Fatemi, 1992]

$$\sum_{i,j}(f_{i,j} - x_{i,j})^2 + \lambda \sum_{i,j} |\nabla x_{i,j}|$$

  - ∇ discrete gradient
  - $\sum_{i,j}|\nabla x_{i,j}|$ discrete total variation (TV)
  - regularization parameter $\lambda > 0$

$\Rightarrow$ edge-preserving

- stair caising-effect: reduced by adding higher order derivatives

Recently [Cremers,Strekalovski, 2012], [Lellmann et al., 2013], [Weinmann et al., 2013], [Bačák, 2013]

- TV denoising generalized to Riemannian manifolds
- several algorithms to find the minimizer x^*
- convergence for CAT(0) spaces
- $\mathbb{S}^1$ is not a CAT(0) space

First & Second Order Differences on $\mathbb{R}$

A short reminder.

Given a weight $w = (w_j)_{j=1}^d \in \mathbb{R}^d \setminus \{0\}$, i.e. $\langle w, 1_d \rangle := \sum_{j=1}^{d} w_j = 0$,

the finite difference operator is given by

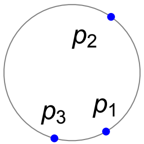
$$\Delta(x; w) := \langle x, w \rangle, \quad x \in \mathbb{R}^d$$

Examples

- $b_1 := (-1, 1)$: First order difference $\Delta(x; b_1) = x_2 - x_1$
- $b_2 := (1, -2, 1)$: Second order difference $\Delta(x; b_2) = x_1 - 2x_2 + x_3$
- $b_{1,1} := (-1, 1, 1, -1)$: ‘mixed second order difference’ $\Delta(x; b_{1,1}) = -x_1 + x_2 + x_3 - x_4$

First & Second Order Difference on $\mathbb{S}^1$

Defined by looking at different situations on $\mathbb{R}$ the points may take.



- $x_i \in [-\pi, \pi) \Leftrightarrow p_i := (\cos x_i, \sin x_i)$
- Idea: unwrap the circle onto any tangential line

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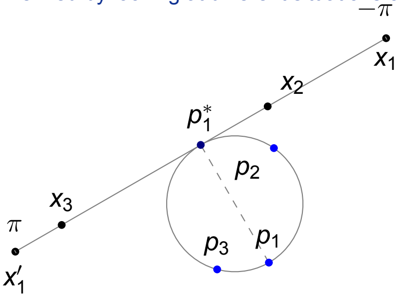

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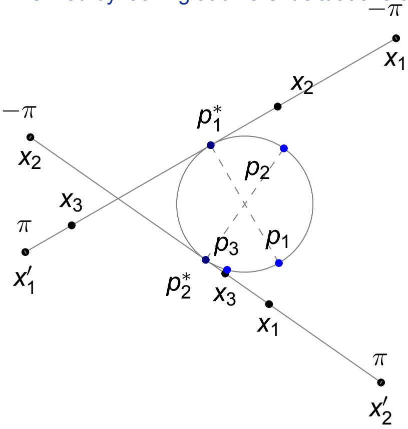

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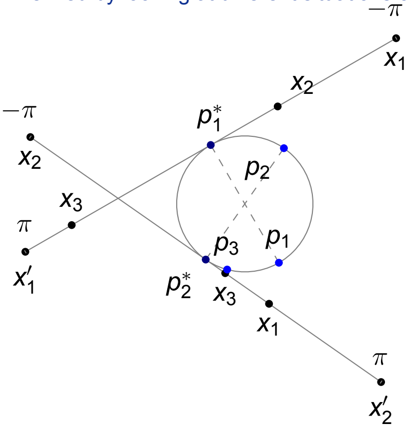

- $x_i \in [-\pi, \pi) \Leftrightarrow p_i := (\cos x_i, \sin x_i)$
- Idea: unwrap the circle onto any tangential line
- Absolute cyclic differences w.r.t w:

$$d(x; w) := \min_{\alpha \in \mathbb{R}} \left| \Delta\left([x + \alpha 1_d]_{2\pi}; w\right) \right|$$

- $[x]_{2\pi}$: element-wise mod 2π
  except $x_i = (2k+1)\pi$: take both $\pm\pi$

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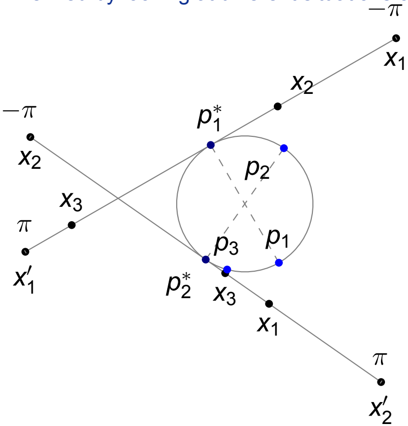

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- b_1: arc length distance $d_1(x_1, x_2) := d(x; b_1) = \left| \left(\Delta(x; b_1)\right)_{2\pi} \right|$
- b_2: $d_2(x_1, x_2, x_3) := d(x; b_2) = \left| \left(\Delta(x; b_2)\right)_{2\pi} \right|$ (similar for $b_{1,1}$)

Second Order Total Variation on the Circle

Transfer the ROF functional to the circle and extend to second order differences.

Let $f = (f_i)_{i=1}^N$ be given data on $\mathbb{S}^1$, $\alpha, \beta \geq 0$.
We are interested in the minimizers $x^* \in (\mathbb{S}^1)^N$ of

$$\varphi(x) := F(x; f) + \alpha\,\mathrm{TV}_1(x) + \beta\,\mathrm{TV}_2(x),$$

where

- data fidelity term $F(x; f) = \frac{1}{2}\sum_{i=1}^{N} d_1(f_i, x_i)^2$
- first order differences $\mathrm{TV}_1(x) = \sum_{i=1}^{N-1} d_1(x_i, x_{i+1})$
- second order differences $\mathrm{TV}_2(x) = \sum_{i=2}^{N-1} d_2(x_{i-1}, x_i, x_{i+1})$
- similar with both drectional differences and a second order mixed derivative for 2D data.

Proximal Point Algorithms on $\mathbb{R}$

The proximal mapping is defined by

$$\operatorname{prox}_{\lambda\varphi}(f) := \operatorname*{arg\,min}_{x\in\mathbb{R}^N} \frac{1}{2}\|f-x\|_2^2 + \lambda\varphi(x),$$

- $\varphi : \mathbb{R}^N \to (-\infty, +\infty]$ proper, closed, convex function
- $\lambda > 0$ trade-off parameter: minimizing $\varphi(x)$ vs. “staying near” f

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Proximal Point Algorithm (PPA): Picard Iteration [Moreau, 1965; Rockafellar, 1976]

$$x^{(k+1)} = \operatorname{prox}_{\lambda\varphi}(x^{(k)}), \quad k \in \mathbb{N}$$

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For $\varphi = \sum_{i=1}^{c} \varphi_i$, where the proximal mappings of φ_i are "easier":

Cyclic Proximal Point Algorithm (CPPA) [Bertsekas, 2011]

$$x^{(k+\frac{i+1}{c})} = \operatorname{prox}_{\lambda_k\varphi_i}(x^{(k+\frac{i}{c})}), \quad i = 0,\ldots,c-1, k \in \mathbb{N}.$$

Converges to a minimizer if $\{\lambda_k\} \in \ell_2(\mathbb{Z})\backslash\ell_1(\mathbb{Z})$

Proximal Mappings on $\mathbb{S}^1$

For cyclic data: $\operatorname{prox}_{\lambda\varphi}(g) = \operatorname*{arg\,min}_{x\in[-\pi,\pi)^N} \frac{1}{2} d_1(g,x)^2 + \lambda\varphi(x)$ [Ferreira, Oliveira, 2002]

Theorem I [B., Laus, Steidl, Weinmann, 2014]

The unique minimizer x^* of $\operatorname{prox}_{\lambda d_1(f,\circ)^2}(g)$ is

$$x^* = \left(\frac{g+\lambda f}{1+\lambda} + \frac{\lambda}{1+\lambda} 2\pi v\right)_{2\pi}, \quad v = \begin{cases} 0 & \text{for } |g-f| \leq \pi, \\ \operatorname{sgn}(g-f) & \text{for } |g-f| > \pi \end{cases}$$

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Theorem II [B., Laus, Steidl, Weinmann, 2014]

The minimizers of $\operatorname{prox}_{\lambda d(\circ;w)}(g)$, $\quad w \in \{b_1, b_2, b_{1,1}\}$, are given by

$$x^* = \left(g - \operatorname{sgn}([\langle g,w\rangle]_{2\pi}) \cdot \min\left\{\lambda, \frac{|(\langle g,w\rangle)_{2\pi}|}{\|w\|_2^2}\right\} w\right)_{2\pi}$$

For $|(\langle g,w\rangle)_{2\pi}| = \pi$, there are two minimizers, otherwise it is unique.

CPPA with Second Order TV for 1D Data on $\mathbb{S}^1$

How to split the higher order TV functional φ?

- $F(x; f) = \frac{1}{2}\sum_{i=1}^{N} d_1(f_i, x_i)^2 =: \varphi_1(x)$
  proximal mapping I (applied simultaneously element-wise)
- first order differences

$$\alpha\,\mathrm{TV}_1(x) = \alpha \sum_{i=1}^{N-1} d_1(x_i, x_{i+1})$$

- second order differences

$$\beta\,\mathrm{TV}_2(x) = \beta \sum_{i=2}^{N-1} d_2(x_{i-1}, x_i, x_{i+1})$$

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$$\alpha\,\mathrm{TV}_1(x) = \sum_{l=0}^{1}\alpha\sum_{i=1}^{\left\lfloor\frac{N-1}{2}\right\rfloor} d_1(x_{2i-1+l},x_{2i-l}) =: \sum_{l=0}^{1}\varphi_{2+l}$$

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  inner sum: distinct data $\Rightarrow$ proximal mapping II with $w = b_1$
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  inner sum: distinct data $\Rightarrow$ proximal mapping II with $w=b_2$

CPPA with Second Order TV for 1D Data on $\mathbb{S}^1$

How to split the higher order TV functional φ?

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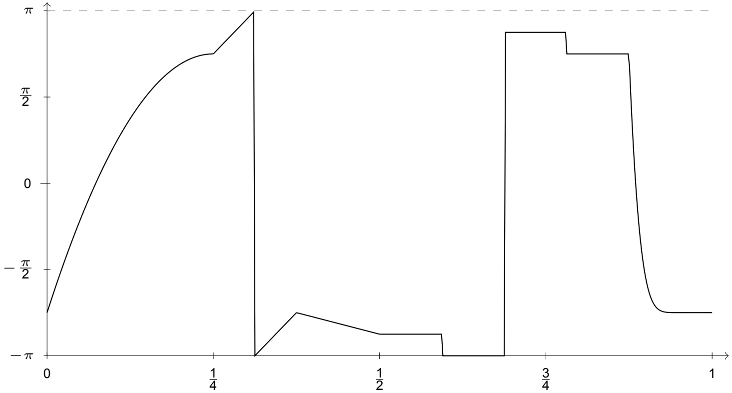
inner sum: distinct data $\Rightarrow$ proximal mapping II with $w = b_2$

$\Rightarrow$ $\varphi(x) = \sum_{l=1}^{6} \varphi_l(x)$ $\Rightarrow$ cycle length $c = 6$

Example

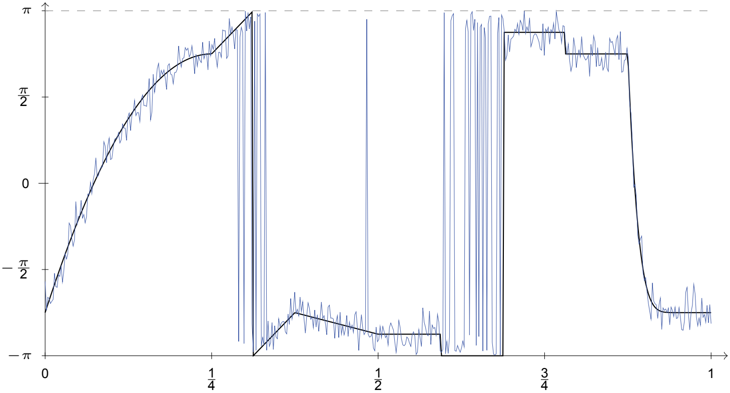
Denoising a 1D phase valued signal.



- function $f\colon [0,1] \to \mathbb{S}^1$ sampled to obtain data $f_\mathrm{o} = (f_{\mathrm{o},i})_{i=1}^{500}$
- jumps $> \pi$ at $\frac{5}{16}$ and $\frac{11}{16}$ are due to the representation system

Example

Denoising a 1D phase valued signal.



- function $f\colon [0,1] \to \mathbb{S}^1$ sampled to obtain data $f_o = (f_{o,i})_{i=1}^{500}$
- adding wrapped Gaussian noise, $\sigma = 0.2$
- noisy data $f_n = (f_0 + \eta)_{2\pi}$

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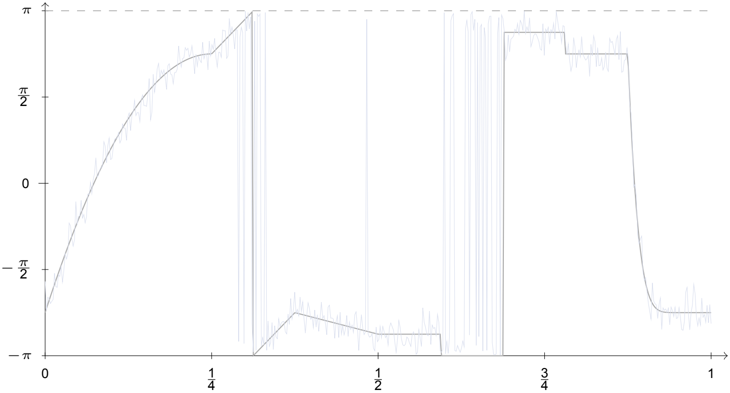

- comparison of f_o & f_n with

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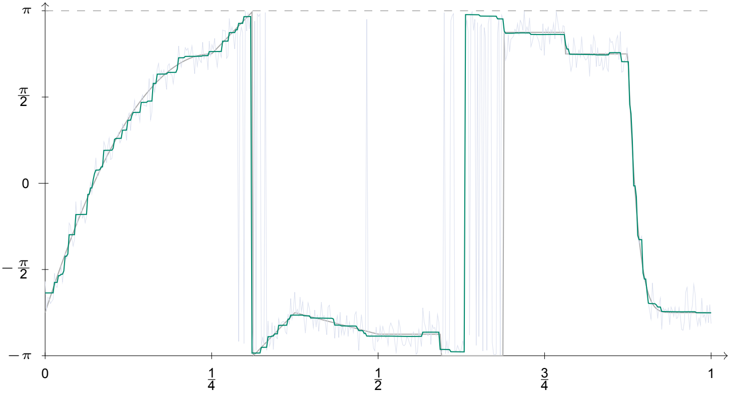

- comparison of f_o & f_n with f_1
- denoising: just TV_1: $\alpha = \frac{3}{4}$, $\beta = 0$
- but: stair casing

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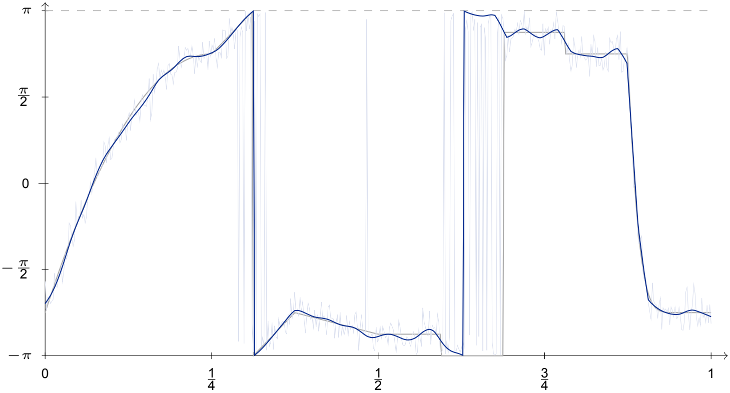

- comparison of f_o & f_n with f_2
- denoising: just TV_2: $\alpha = 0$, $\beta = \frac{3}{2}$
- but: no plateaus

Example

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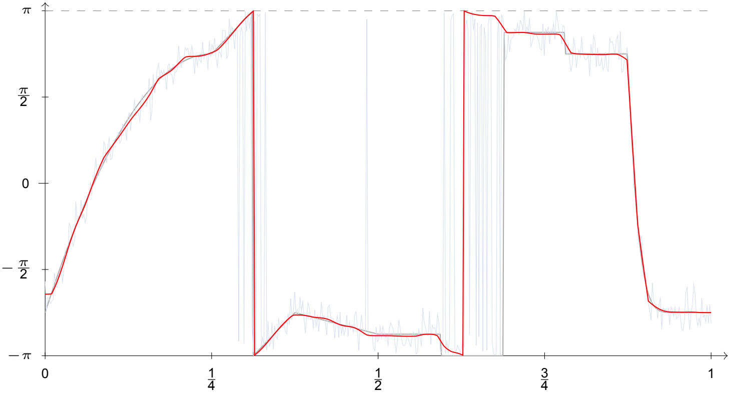
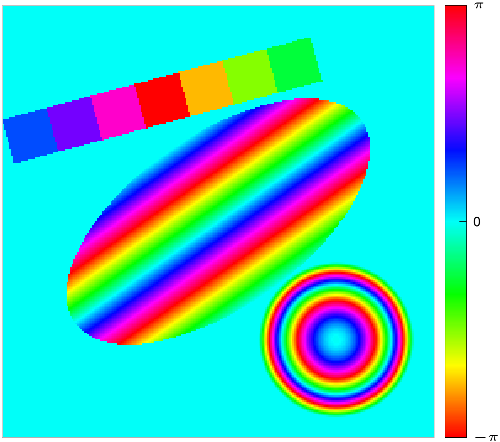


- comparison of f_o & f_n with f_3
- denoising: TV_1 & TV_2: $\alpha = \frac{1}{2}$, $\beta = 1$
- smallest mean squared error

Example

Denoising artificial phase valued data.



original data f_o, 256×256 pixel image

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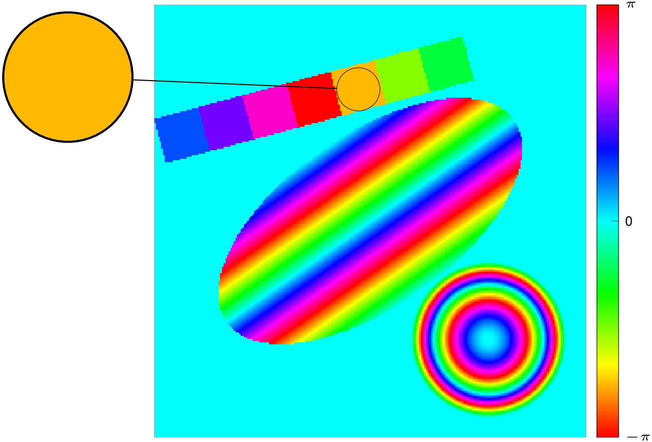

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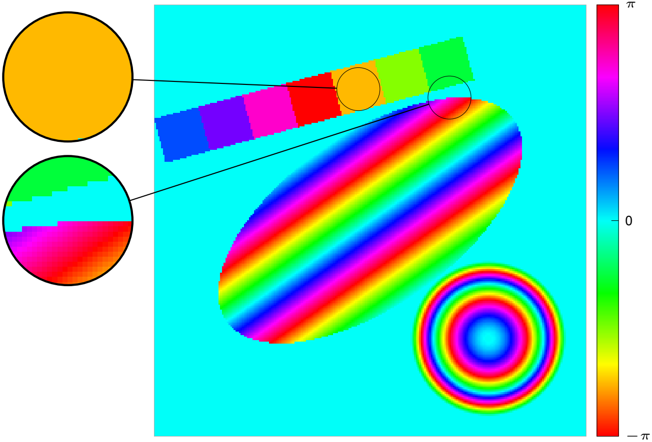

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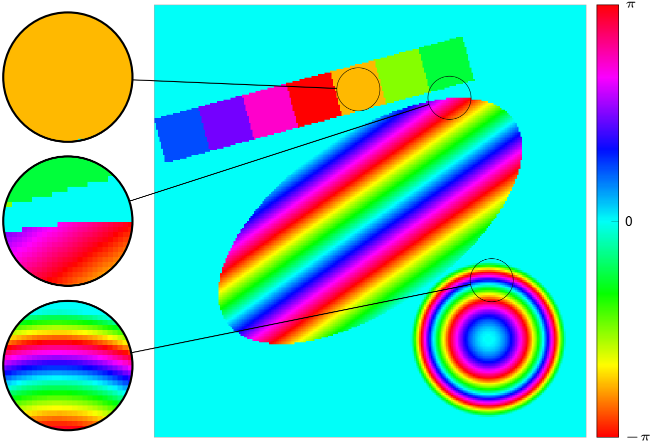

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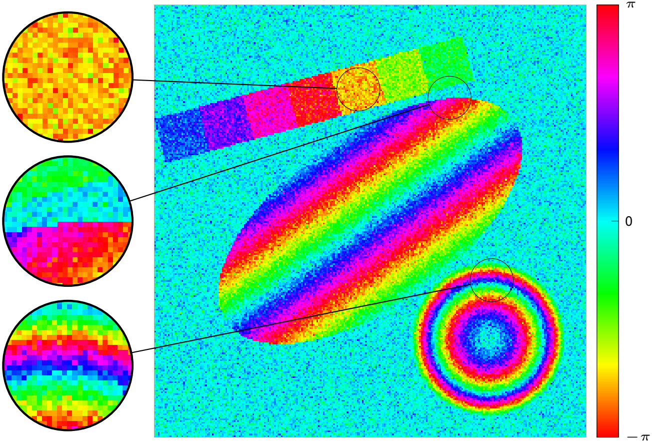

noisy data f_n, $\sigma = 0.3$

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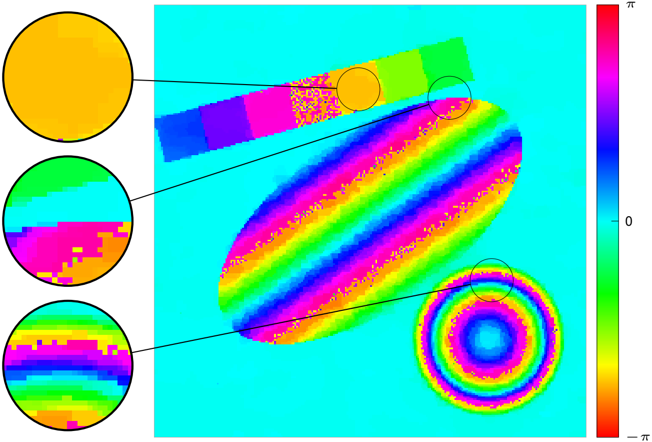

denoising with real valued TV
$\alpha_1 = \frac{3}{8}$, $\alpha_2 = \frac{1}{4}$: red part destroyed.

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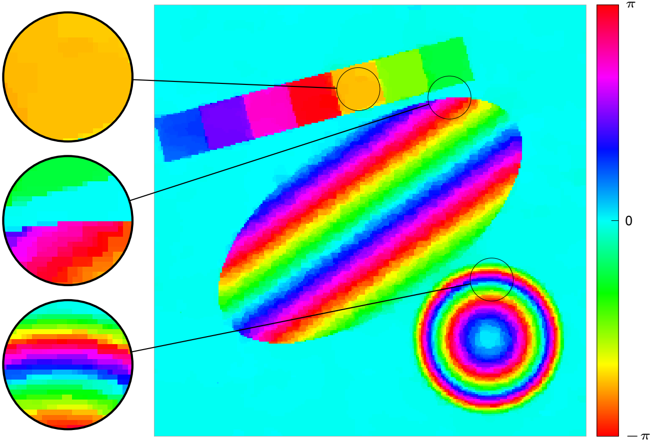

denoising f_n: f_1 with just TV_1

$\alpha_1 = \frac{3}{8}$, $\alpha_2 = \frac{1}{4}$, $\beta_1 = \beta_2 = \gamma = 0$: stair casing

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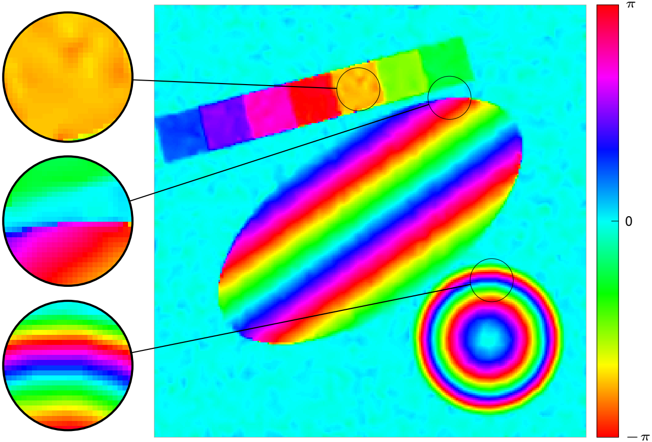
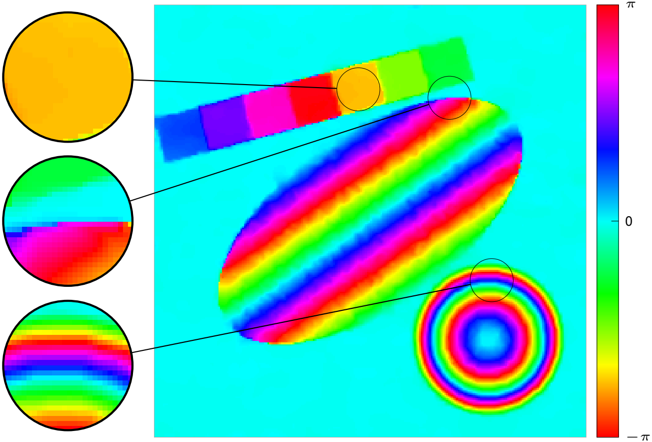


denoising f_n: f_2 with just TV_2

$\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = \gamma = \frac{1}{8}$: no plateaus

Example

Denoising artificial phase valued data.

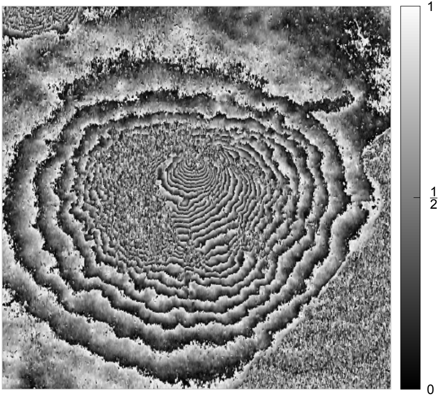


denoising f_n: f_3 with TV_1 & TV_2

$\alpha_1 = \frac{1}{4}$, $\alpha_2 = \beta_1 = \beta_2 = \frac{1}{8}, \gamma = 0$: smallest mean squared error

Mount Vesuvius

The following image is InSAR data from Mount Vesuvius, Italy.[1]



original data, 432×426 pixel

[1] https://earth.esa.int/workshops/ers97/program-details/speeches/rocca-et-al/

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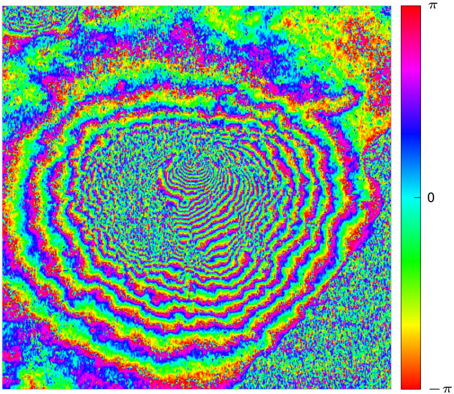

adapted just the coloring

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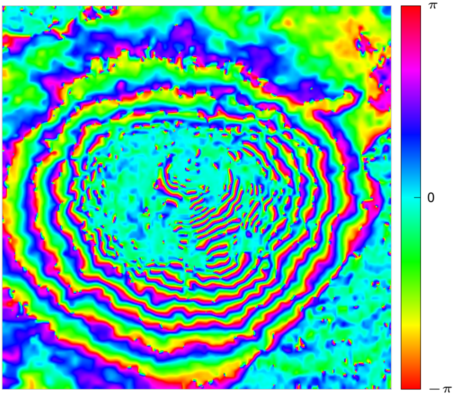

denoised: $\alpha_1 = \alpha_2 = \frac{1}{4}, \beta_1 = \beta_2 = \gamma = \frac{3}{4}$

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Summary

We derived for $\mathbb{S}^1$-valued 1D & 2D data f

- higher order differences
- proximal mappings for first and second order differences
- higher order TV functional φ
- an efficient CPPA to minimize φ
- applications:
  - InSAR data denoising
  - hue denoising
  - habituation data
  - Fourier optics, ground based astronomy

We can proof convergence of the algorithm to a minimizer under certain conditions, which are due to the nonconvexity of φ.

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Thank you for your attention.