The multivariate anisotropic Wavelet Transform on the Torus

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Introduction

Wavelets: fast decomposition of a function or signal into “levels of detail”.
here: periodic function(s)

one-dimensional

- shift invariant spaces based on the shift $2\pi/N$, $N \in \mathbb{N}$
- periodic wavelets, e.g. in [PT95]
- characterizations and fast algorithm in freuency domain

d-dimensional

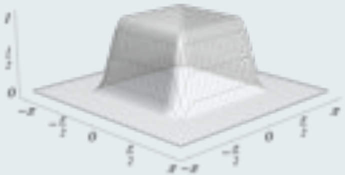
- “curse of dimension”
- reduced by focusing on certain directions [LP10, MS03]
- develop corresponding wavelets towards adaptivity/focus
- adapt characterizations and fast algorithms in frequency domain

+ characterize directions

Sampling

using patterns to specify certain directions of interest

Let's say our data is given by certain measurements of a 2π periodic d-dimensional function $f\colon \mathbb{T}^d \to \mathbb{R}$ (or $\mathbb{C}$),



Example: a simple "hill" ($d = 2$)

more preciesly by $2\pi\mathbf{y}, \mathbf{y} \in \mathcal{P}(\mathbf{M})$, where $\mathbf{M} \in \mathbb{Z}^{d\times d}$ is invertible and

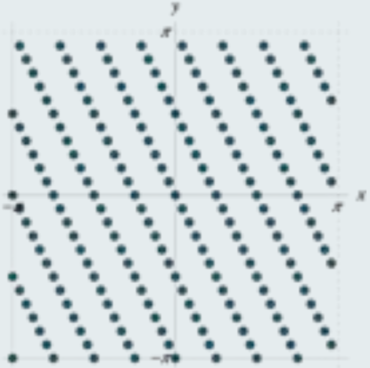
$$\mathcal{P}(\mathbf{M}) := \mathbf{M}^{-1}\mathbb{Z}^d \cap \Big[-\frac{1}{2}, \frac{1}{2}\Big)^d$$

- specify "directions of interest" (columns of $\mathbf{M}^{-1}$, the inverse of $\mathbf{M}$)
- $m = \big|\mathcal{P}(\mathbf{M})\big| = |\det \mathbf{M}|$ number of points
- How to get information about "detail" in one of these directions?
  Idea: More points in directions of interest

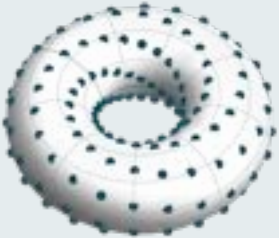
The pattern $\mathcal{P}(\mathbf{M})$

Sampling with interest in a specific direction

$$\mathbf{M} = \begin{pmatrix} 32 & -8 \\ 8 & 4 \end{pmatrix}$$



The pattern corresp. to **M**

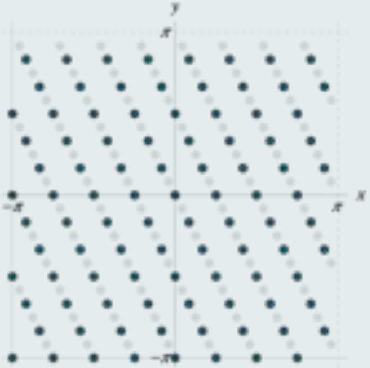


drawn on a torus

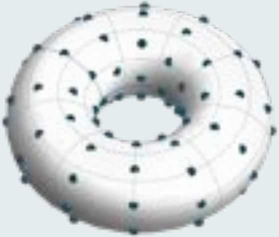
The pattern $\mathcal{P}(\mathbf{N})$

Sampling with less interest in a specific direction

$$\mathbf{M} = \mathbf{J}\mathbf{N} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 16 & -4 \\ 8 & 4 \end{pmatrix}$$



The pattern corresp. to **N**



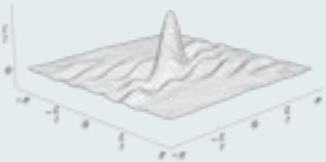
drawn on a torus

Shift Invariant spaces

Approximate *f* by shifts of one function.

Given a second function φ:
Approximate *f* by a sum of translates of φ

$$f(\mathbf{x}) \approx f_{\mathbf{M}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{\mathbf{y}}\varphi(\mathbf{x} - 2\pi\mathbf{y})$$



The Dirichlet kernel $\varphi = D_{\mathbf{N}}$ [LP10]

Set of $a_{\mathbf{y}} \in \mathbb{C}$: “a building plan” for *f*.

necessary: $\{\varphi(\circ - 2\pi\mathbf{y})\}_{\mathbf{y}\in\mathcal{P}(\mathbf{M})}$ are linear independent

The space of all possible funktions of these translates is called the *shift invariant space* of φ with respect to **M**, denoted by

$$V_{\mathbf{M}}^{\varphi} := \operatorname{span}\{\varphi(\circ - 2\pi\mathbf{y}),\ \mathbf{y} \in \mathcal{P}(\mathbf{M})\}.$$

Decomposition

Looking at $\mathbf{M} = \mathbf{J}\mathbf{N}$,
where $\mathbf{N}$ is (still) integer valued and (for simplicity) $|\det \mathbf{J}| = 2$
$\Rightarrow$ decompose

$$V_{\mathbf{M}}^{\varphi} = V_{\mathbf{N}}^{\xi} \oplus W_{\mathbf{N}}^{\psi},$$

where

- construct $\xi \in V_{\mathbf{M}}^{\varphi}$ (analogously to φ before) $\Rightarrow$ compute ψ (the wavelet)
- both $V_{\mathbf{N}}^{\xi}$ and $W_{\mathbf{N}}^{\psi}$ are both shift invariant (w.r.t. $\mathbf{N}$) spaces

Hence, we have a **unique** decomposition for $f_{\mathbf{M}} \in V_{\mathbf{M}}$:

$$f_{\mathbf{M}} = f_{\mathbf{N}} + g_{\mathbf{N}}, \quad f_{\mathbf{N}} \in V_{\mathbf{N}}^{\xi}, \ g_{\mathbf{N}} \in W_{\mathbf{N}}^{\psi},$$

where

- $f_{\mathbf{N}}$ is a (depending on $\mathbf{J}$) certain rougher approximation
- $g_{\mathbf{N}}$ is the difference of both approximations $\Rightarrow$ detail

The (fast) Fourier transform on $\mathcal{P}(\mathbf{M})$

The Fourier matrix with respect to $\mathbf{M}$

$$\mathcal{F}(\mathbf{M}) := \frac{1}{\sqrt{m}} \left(\mathrm{e}^{-2\pi \mathrm{i} \mathbf{h}^T \mathbf{y}} \right)_{\mathbf{h} \in \mathcal{G}(\mathbf{M}^T), \mathbf{y} \in \mathcal{P}(\mathbf{M})} \in \mathbb{C}^{m \times m}$$

where

- $\mathbf{h} \in \mathcal{G}(\mathbf{M}^T) := \mathbf{M}^T \mathcal{P}(\mathbf{M}^T)$ adresses the rows
- $\mathbf{y} \in \mathcal{P}(\mathbf{M})$ the columns
- $\mathbf{a} = (a_{\mathbf{y}})_{\mathbf{y} \in \mathcal{P}(\mathbf{M})} \in \mathbb{C}^m$ (same order als the colums): DFT on $\mathcal{P}(\mathbf{M})$

$$\hat{\mathbf{a}} = (\hat{a}_{\mathbf{h}})_{\mathbf{h} \in \mathcal{G}(\mathbf{M}^T)} = \sqrt{m} \mathcal{F}(\mathbf{M}) \mathbf{a} \in \mathbb{C}^m$$

$\Rightarrow$ fast Fourier transform for special ordering [Be13], i.e. in $O(m \log m)$

- Sample f on $\mathcal{P}(\mathbf{M})$ to compute $f_{\mathbf{M}} \in V_{\mathbf{M}}^{\varphi}$ (in Fourier domain)

$\Rightarrow$ decomposition (using Parseval equality on $L_2(\mathbb{T}^d)$) in $O(m)$

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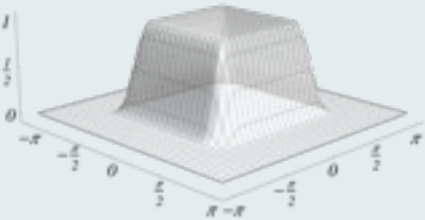
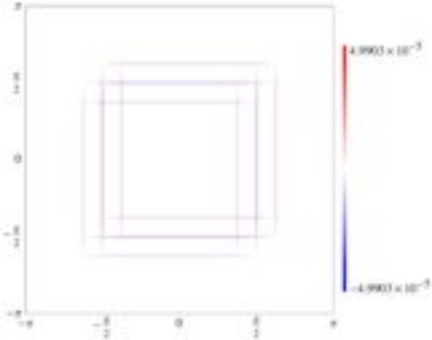
Example

search for changes of steepness in the “hill” - sampling

- $\mathbf{M} = \begin{pmatrix} 512 & 0 \\ 0 & 512 \end{pmatrix} \Rightarrow$ “classical” sampling
- $\varphi = D_{\mathbf{M}}, \xi = D_{\mathbf{N}}$, the Dirichlet kernels [LP10]



the “hill” from the introduction
a Box spline

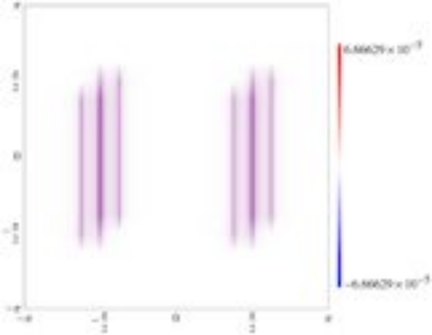


the error $f - f_{\mathbf{M}}$

Example

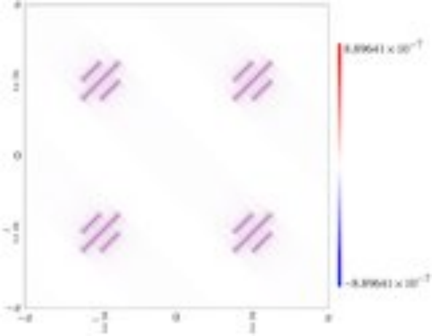
search for changes of steepness in the "hill" - decompose $f_{\mathbf{M}}$

$\mathbf{J} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$



the decomposition in order to detect in horizontal direction

$\mathbf{J} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

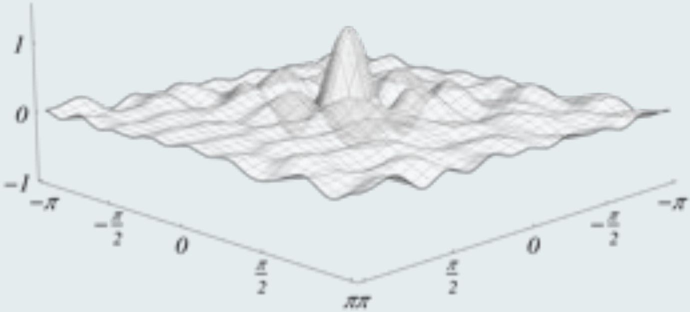


the decomposition in order to detect in both diagonal directions

Current work

The Wavelet should be

- localised
- still easy to compute (in Fourier coefficients)
- only dependent on **J**

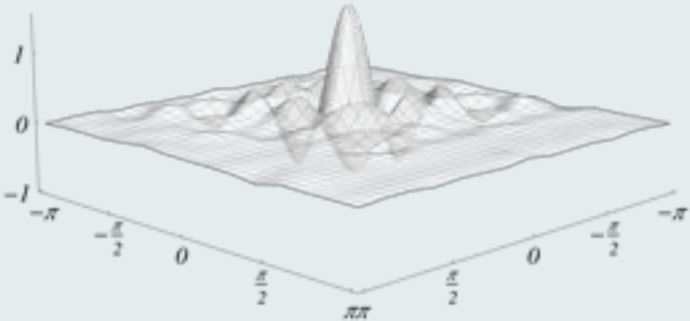


Dirichlet kernel Wavelet

Current work

The Wavelet should be

- localised
- still easy to compute (in Fourier coefficients)
- only dependent on **J**



A more localized wavelet "for the same directions"

Summary

In order to obtain directional information about (sampled data of) *f*

- Choose φ, ξ (depending on "direction" $\mathbf{J}$), such that
  - they fulfill the requirements (w.r.t. $\mathbf{M}$ resp. $\mathbf{N}$)
  - their translates approximate *f* well (at least w.r.t. certain criteria)
  - neccessary: $\xi \in V_{\mathbf{M}}^{\varphi}$

  $\Rightarrow$ translates of wavelet ψ model the detail
- further decomposions of $\mathbf{N}$ $\Rightarrow$ Multiscale Analysis
- using the fact, that $f_{\mathbf{M}}, \xi, \psi \in V_{\mathbf{M}}^{\varphi}$

  $\Rightarrow$ all computations possible by using "$\{a_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{P}(\mathbf{M})}$-type" data

  $\Rightarrow$ using their Fourier series and the discrete FourierTransforms

  $\Rightarrow$ fast algorithms in $O(m)$

Thanks for your attention.

...and of course the organizers for this nice workshop.

Literature

[Be13] rb, *The fast Fourier Transform and fast Wavelet Transform for Patterns on the Torus*, to appear in ACHA (2013). doi: 10.1016/j.acha.2012.07.007
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[LP10] D. Langemann, J. Prestin, *Multivariate periodic wavelet analysis*, ACHA 28 (2010) 46–66. doi: 10.1016/j.acha.2009.07.001

[MS03] I. E. Maximenko, M. A. Skopina, *Multivariate periodic wavelets*, St. Petersbg. Math. J. 15 (2003) 165–190. doi: 10.1090/S1061-0022-04-00808-8

[PT95] G. Plonka, M. Tasche , *On the computation of periodic spline wavelets*, ACHA 2 (1995) 1–14. doi: 10.1006/acha.1995.1001